On Frege's True Way Out (Paper), K50-Set8b

Werner-Jimmy De Pauli-Schimanovich[1]

On Frege´s true way out

(*Half-page abstract*)

The lecture introduces a new system of set theory called FNM (=Formalized Naive Mengenlehre). In the usual naive set theory, the (extensional) Principle of Abstraction ($P o A$) $x \in \{y{:}\varphi(y)\} \Leftrightarrow \varphi(x)$ allows the contextual replacement of the set operator. In FNM the set operator is a basic notion and the ($P o A$) is replaced by the new (metatheoretical or intensional) True Abstraction Principle

$$(TAP) \quad x \in \{y{:}\varphi(y)\} \Leftrightarrow \varphi(x) \wedge \{y{:}\varphi(y)\} \notin^{*} x$$

where the additional term ($\{y{:}\varphi(y)\} \notin^{*} x$), the so called Zusatz, is a metalinguistic abbreviation for a long formula of the (set theoretical) object language. The Zusatz is constructed in such a way that it is satisfied automatically if we want to form "normal" sets $\{y{:}\varphi y\}$, e.g. sets belonging to the commulative hierarchy of ZF. Only by using pathological (not "well-founded") predicates, the Zusatz is becoming false and prevents that way the generation of inconsistencies.

Accordingly, the Zusatz means the following:
The set operator is no element of an n-limbed epsilon-chain of x. The $\in$-chain is defined informally:

$$x \in^{0} y :\underset{def}{=} x = y; \quad x \in^{n+1} y :\underset{def}{=} \bigvee_{z} \cdot x \in z \wedge z \in^{n} y.$$

Furthermore, $x \in^{*} y$ means $\exists n{:}x \in^{n} y$ (for $n \in \omega$). Note that the case $n = 0$ is included and $x \notin^{*} y$ is wrong for $x = y$. The idea of restricting the ($P o A$) was already given by Frege himself, but he used as Zusatz $x \neq \{y{:}\varphi\}$. This was of course to weak to prevent antinomies.

The (TAP) has a trivial model ($\varnothing, \varepsilon$) and is therefore consistent. M. Goldstern pointed out that if we add it to ZF it is inconsistent because the (Sum Axiom) allows to construct pathological sets. But (TAP) is so strong that you don´t need the full power of ZF. We can replace the axioms of ZF by much weaker conditions, e.g. the (very strong) power set axiom of ZF by the selfcontained statement $a \in \mathbb{P}(a)$ (for all sets a). Other additional selfcontained axioms are: $\bigvee_{x} \varphi(x) \rightarrow \{y{:}\varphi y\} \neq \varnothing$, or: $a \in \{a,b\}$. These additional axioms are weaker than ZF; but together with (TAP) they yield a system FNM at least as strong as ZF minus (Sum Axiom) (and relative consistent to it probably.)

(TAP) offers possibility to view set theory under an unique principle and therfore consider it as part of logic. That enables us to put logicism on a new foundation.

[1]Institut für Statistik, OR und Computerverfahren, Austria, Universitätsstraße 5, A-1010 Wien, Fax: +431-40407-88.
Or: Departamento de Informatica, Campus Tafira, Universidad de Las Palmas, E-35.080 Las Palmas de Gran Canaria, Spain. Fax: +34-28-458785.

Werner-Jimmy De Pauli-Schimanovich[1]

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A New Approach to the Formalization of Naive Mengenlehre[2].

This paper introduces a new system of set theory called FNM (= Formalized Naive Mengenlehre) which is on example par exelance that only philosophy-guided investigation can solve the essential problems in mathematics. This was Kurt Gödel's conviction, which in particular led him to his famous results. However it is in opposition to the super-technical research of today. The author solved the 90 year old wish of Georg Cantor and Gottlob Frege of how to formalize (consistently) naive mengenlehre. I could establish the system FNM without a sophisticated technical apparatus by having the philosophical insight about how to solve Frege's problem[3].

FNM can also be considered as justification for the ignorance of the Bourbaki school[4] which ignored the Zermelo-Fraenkel system ZF up to recently, arguing that mathematicians - guided by their intuition - in practice never fail. But from the logical point of view, also, naive mengenlehre turns out to be a sufficiently solid basis for mathematics, as Bourbaki always wanted. This is shown by formalizing naive mengenlehre and establishing the system FNM which works in practice like naive set theory.

In the usual naive set theory, the (extensional) Principle of Abstraction $x \in \{y{:}\varphi(y)\} \leftrightarrow \varphi(x)$ allows the contextual replacement of the set operator. In FNM the set operator is a basic notion and the Principle of Abstraction is replaced by the new (metatheoretical or intensional) True Abstraction Principle

$$\text{(TAP):} \qquad x \in \{y{:}\varphi(y)\} \Leftrightarrow \varphi(x) \wedge \left(\{y{:}\varphi(y)\} = \varnothing \vee \nexists n{:}\{y{:}\varphi(y)\} \in^{n} x\right) \wedge \neg Miri(\{y{:}\varphi y\}).$$

[1]Institut für Statistik, OR und Computerverfahren, Austria, Universitätsstraße 5, A-1010 Wien, Fax: +431-40407-88.
Or: Departamento de Informatica, Campus Tafira, Universidad de Las Palmas, E-35.080 Las Palmas de Gran Canaria, Spain. Fax: +34-28-458785.

[2]The author wants to express his thanks to all collegues who helped in checking the system FNM for consistency and who gave hints for formulating this paper: Maurice Boffa, Norbert Brunner, T.E. Forster, Martin Goldstern, Jaakko Hintikka, Ronald B. Jensen, Yehuda Rav and Dana Scott. Special thanks are due for the Collegium Logicum Vindobonensis and the friends of the Kurt Gödel Society, especially to: Matthias Baaz, Norbert Brunner, Georg Gottlob, Eckehart Köhler, Alexander Leitsch and Franz Pichler, for encouraging the author to write this paper. Finally I want to take the occassion to thank my teachers Curt Christian, Edmund Hlawka and Leopold Schmetterer who advised me on the way to mengenlehre.

[3]Gottlob Frege: "Grundgesetze der Arithmetik", vol.2, p.262 ff., English Translation: Peter Geach & Max Black (Eds. & Transl.): "Translations from the Philosophical Writings of Gottlob Frege", Blackwell Publ., Oxford 1952, p.242 ff.

[4]Cf: A.R.D. Mathias: "The Ignorance of Bourbaki", The Mathematical Intelligence 14 (No.3), pp.4-13, (1992); the discussion about Zermelo's set theory, p.6 (.......), p.12.

where the additional term $(\{y{:}\varphi y\} = \varnothing \vee \nexists n{:}\{y{:}\varphi y\} \in^n x)$, the so called Zusatz, is a metalinguistic abbreviation for a long formula of the (set theoretical) object language.

The Zusatz is constructed in such a way that it is satisfied automatically if we want to form "normal" sets $\{y{:}\varphi(y)\}$ and their elements x (e.g. sets belonging to the commulative hierarchy of ZF), and since it is true for "normal" sets, the Zusatz can be ignored as a redundant part of the formula (TAP). This means: For normal sets (respectively, for constructible sets from the commulative hierarchy), the extensional Principle of Abstraction of the naive mengenlehre is valid in FNM, too, in the same way as we have used it in ZF until now. Therfore FNM justifies the procedural manner of Bourbaki. Only by using pathological (not well-founded) predicates (which in fact we should not consider at all), the Zusatz is becoming false and prevents that way the generation of inconsistencies.

Accordingly, the Zusatz means the following: The set operator is either empty or it is no element of an n-limbed epsilon-chain of x (for $n \in \omega$). The $\in$-chain is defined recursivly and informaly as follows:

$$x \in^0 y :=_{def} x = y; \quad x \in^{n+1} y :=_{def} \bigvee_z \cdot x \in z \in^n y.$$

The right side of the Zusatz (and even the existential quantification of this abbreviation) can be defined in the object language in the following way:

$$\exists n{:} x \in^n y :=_{def} \quad x = y \quad \dot\vee$$
$$\dot\vee \bigvee_n \mathrm{Nat}(n) \neq \varnothing \wedge \bigvee_f \left(\mathrm{Function}(f) \wedge \mathrm{Domain}(f) = n+1 \wedge f(\varnothing) = x \wedge f(n) = y \wedge \bigwedge_{i \in n} (f(i) \in f(i+1))\right)$$

Here "Function (f)" means that the set f has the functional property; "Domain $(f) = n+1$" means that f is a function from the domain $n+1 = \{0,1,2,\ldots,n-1,n\}$ into an arbitrary set; and $f(x)$ is the function value of f defined in the usual way (in ZF or NBG).

Mathematics treats sets as predicate-extensions and the Principle of Abstraction is correctly used therfore. But the mistake that has been made (for nearly a century) was to make a dogma out of this use, i.e. that sets exclusivly and always have to be predicate-extensions. That caused the rise of antimomies. In FNM, a set exists for arbitrary predicates φ, but for pathological predicates, their extensions are limited to the corresponding meaning of the predicate and the sense intended in its construction.

Thus, for example, Russell´s set is the set of all sets not containing themselves as elements, except the set itself, and except all "derivated" set terms constructed on the basis of it. With this principle of intensional set comprehension based on the intended meaning of the predicate $x \notin x$, no contradiction can arise by checking whether Russell-set ru is its own element or not:

$$ru \in ru \Leftrightarrow ru \notin ru \wedge (ru = \varnothing \vee \nexists n{:} ru \in^n ru).$$

Since the Zusatz is wrong in this statement, it follows that $ru \notin ru$. Similarly, by falsifying the Zusatz, the other antiomies are prevented, too, and hence no contradiction could be derived form FNM until now[5].

The Zusatz to the (extensional) Principle of Abstraction converts it to the (intensional) True Abstraction Principle which (in the normal case of constructing ZF-sets) is verifiable and therefore redundant: one can use the (TAP) like the Principle of Abstraction of the naive mengenlehre (i.e. ignore the Zusatz). But since we do not have any criterion for normality of predicates, we do not restrict the general use of predicates, but prevent antinomies by making the Zusatz false. Thus the Zusatz solves our problem; nevertheless it has a great disadvontage: it makes the (TAP) in an impredicativee equivalence that cannot be used as a definition or as a set-comprehension schema, since the object to be defined (i.e. the set term on the left hand side) appear again on the right hand side of the equivalence.

This impredicativity of (TAP) is philosophically inconvenient but does not cause major technical difficulties because the self-refference of the set term can be resolved as in a recursion. Starting from x in the Zusatz, one can climb down the $\in$-ladder step by step until he ends either at $\varnothing$ (after m steps, thus satisfying the Zusatz, because the set term cannot be an element of the (n-m)-th step of $\varnothing$ has no elements at all), or one ends at some pathological set (which falsifies the Zusatz according to its definition). In a third case, the Zusatz need not be evaluated because $\varphi(x)$ is false.

Furthermore, since the set term in the Zusatz stands on the left side of the $\in$ (whereas it is on the right side of the $\in$ which is on the left hand side of the equivalence which defines the set term), the extensional size of the set is fixed uniquely by TAP in most cases. It is interesting that omitting the impredicative use of the set operator (and letting only its element x appear impredicatively in the Zusatz), one cannot avoid the antinomies. If we take Fund (x) or $\not\exists\infty{:}\ldots,\in^{\infty} x$ or $\not\exists n{:}x \in^{n} x$ or etc as the Zusatz in TAP, new antinomies again become derivable (e.g. Mirimanoff).

The idea of restricting the Principle of Abstraction was already given by Frege himself in the Appendix to his famous Grungesetze der Arithmetik[6] (Vol.2) where he suggests as a revised schema:

$$(*)\quad x \in \{y{:}\varphi(y)\} \leftrightarrow \varphi(x) \wedge x \neq \{y{:}\varphi(y)\}.$$

This is of course not enough to prevent antinomies in general (but only the Russell paradox); this was shown by Willard Van Orman Quine in his paper "On Frege´s Way Out"[7] and strenghtened by Peter Geach in a discussion note with the same title[8]. A modification of $(*)$

[5]The author is working together with his collegues on a consistency proof of FNM relative to ZF.

[6]We also do not want to consider stratification of formulas as a criterion for normality as it is used in Quine´s system New Foundation NF.

[7]Willard V. Quine, "On Frege´s Way Out", Mind vol.64, 1955, pp.145-159.

[8]Peter Geach, "On Frege´s Way Out", Mind vol.65, 1956, pp.408&409.

was given by Jaakko Hintikka[9] (using a suggestion of Wittgenstein) interprets different bound variables exclusively (i.e. as different individuals). But "As a matter of fact, no further attempt was made since 1957 to continue along these lines." as Abraham Adolf Fraenkel and Yoshua Bar-Hillel stated in their book "Foundations of Set Theory"[10]. In fact, the next attempt was my Ph.D. theses, published in a literary journal[11] in 1971. It already containded the present idea of the Zusatz (formulated in terms of the transitive $\in$-closure), but had a technical mistake. Now in this paper the right philosophical treatment (to prevent antinomies)) is also solved correctly from a technical standpoint.

We should also keep in mind that FNM is a Boolean lattice (since, to every set x, its complement $\bar{x}$ exists as a set), and that FNM can easily be used as a basis for category theory, since the universal set $vo = \{x : x = x\}$ (the volle Menge) forms a set. Furthermore the use of naive mengenlehre with the modified Principle of Abstraction (TAP) is a partial justification of logicism and the rehabilitation of its program.

After this long philosophical and historical introduction, it is time to start building up FNM technically and in detail. The system is based on calss variables $A, B, C, \ldots$, where $X, Y, Z, \ldots$ can also be quantified over. For the notion of set we use the usual "... is a set" or $\mathrm{Set}\,(X) := \bigvee_{y}(X \in Y)$. Set variables should be lowercase letters $a, b, c, \ldots, x, y, z, \ldots$ We use equality and the axiom of extensionality

$$(E)\; A = B \leftrightarrow \bigwedge_x (x \in A \leftrightarrow x \in B).$$

The class operator constitutes classes $\{|x : \varphi(x)|\}$ for which the old Principle of Abstraction holds: $x \in \{|y : \varphi y|\} \leftrightarrow \varphi(x)$, because classes are in fact predicate-extensions.

One should not mix this up with the set operator $\{x : \varphi(x)\}$, which comprehends only predicate-intensions (which may be different from the extensions in the case of pathological predicates). The set operator is an inaliminable primitive notion of FNM and produces a set $\{x : \varphi(x)\}$ for every predicate φ (in general and without restrictions). It is strong enough to fill up the full comulative hierarchy of ZF and it generates a much larger universe than $\overset{\iota}{\ldots}_{ZF}$. This way, set construction can be served without weakening the process set comprehension. Its correct formulation as an axiom schema is:

$$(\{:\ldots\})\; \forall \text{ properties } \varphi : \{x : \varphi(x)\} \text{ is a set.}$$

[An alternative formulation would be $\bigvee_x \cdot\, x = \{y : \varphi(y)\}$.]

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[9]Jaakko Hintikka, "Identity, Variable and Impredicative Definitions", J.S.L21, pp.225-245, and "Vicious Circle Principle and the Paradoxes", J.S.L.22, pp.245-249.
[10]Abraham A. Fraenkel & Yoshua Bar-Millel, "Foundations of Set Theory", Springer Verlag, Berlin/N.Y., 2nd edition, 1973.
[11]Werner Schimanovich, "Der Mengenbildungs-Prozess", Manuskripte 33/'71, Editors: Alfred Kolleritsch and Günther Waldorf, Forum Stadtpark, A-8010 Graz, Austria.

As the next axiom we have the already widely discussed intensional principle (TAP) which regulates the element hood of sets on the left-hand side of the set operator. Like the preceding axiom it is a schema, too:

$$\text{(TAP)}\ \forall\varphi{:}\, x \in \{y{:}\,\varphi y\} \Leftrightarrow \varphi x \wedge (\{y{:}\,\varphi y\} = \varnothing \vee \cancel{\exists} n{:}\, \{y{:}\,\varphi y\} \in^{n} x) \wedge \neg \mathrm{Miri}(\{y{:}\,\varphi y\}).$$

In the Zusatz there are a lot of special sets from which we do not know whether they exist a priori or not (because we decide their size by means of themselves with the Zusatz): for example natural numbers, successor of a number, ordered pairs (with a number in their first position), and functions (with a natural number as domain). The following definitions and existence axioms guarantee that the corresponding sets always exist, so that the Zusatz in TAP can always be decided (if it is true for given φ and x or not.) They also ensure that, in particular, the bounded variables n and f in the Zusatz are always sets. It is worth while to mention that our definition of natural numbers (Def Nat) may also allow pathological numbers $\{\varnothing, a\}$ with $a = \{a\}$. But surprisingly, we can show in FNM that $a \neq \{a\}$. Another possibility is that we assume the Axiom of Foundation, which prohibits $a = \{a\}$.

Thus we conclude: The Zusatz can be formulated in the object language and decided (for every φ and x) by means of the following definitions and axioms of FNM:

(Def Nat)

$$Nat(x){:}\Leftrightarrow x = \varnothing \vee \left(\varnothing \in x \wedge \bigwedge_{y=0} y \in x \rightarrow \bigvee_{z\in x} \bigwedge_{u} (u \in y \leftrightarrow u \in z \vee u = z)\right).$$

$$(\exists n+1)\ \mathrm{Nat}(x) \wedge \bigwedge_{y} (y \in Z \leftrightarrow y \in x \vee y = x) \rightarrow \mathrm{Set}(Z).$$

For every natural number the successor is a set.

(Def$\langle n,.\rangle$)

$$x \in \langle n,a\rangle{:}\Leftrightarrow \bigwedge_{y} (y \in x \leftrightarrow y = n) \vee \bigwedge_{y} (y \in x \leftrightarrow y = n \vee y = a).$$

$$(\exists\langle n,.\rangle)\ \mathrm{Nat}(n) \wedge \mathrm{Set}(X) \rightarrow Set(\langle n,X\rangle).$$

For every natural number n, the ordered pair with n in its first position is a set.

(Def Domain)

$$\mathrm{Domain}(F,n){:}= \bigwedge_{x}\left(x \in F \leftrightarrow \bigvee_{m}\bigvee_{y} \cdot x = \langle m,y\rangle \wedge m \in n \right).$$

(Def Function)

$$\mathrm{Function}(F){:}\Leftrightarrow \bigwedge_{m}\bigwedge_{x}\bigwedge_{y} (\langle m,x\rangle \in F \wedge \langle m,y\rangle \in F \rightarrow x = y).$$

$$(\exists\ \mathrm{Function}{:}\, n \rightarrow \ldots) \bigwedge_{n}\bigwedge_{F} \mathrm{Nat}(n) \wedge \mathrm{Function}(F) \wedge \mathrm{Domain}(F,n) \rightarrow Set(F)$$

The functions from a natural number into an arbitrary class are sets. This block of axioms, together with E, $\{\ldots\}$ and TAP, form the basic system of FNM.

If we substitute the set comprehending properties of ZF (i.e.
$x \neq x,\ x = a \vee x = b, \bigvee_{y} \cdot x \in y \wedge y \in a, x \subseteq a, x \in a \wedge \varphi(x), x = \varnothing \vee \left[\varnothing \in x \wedge \bigwedge_{y \neq 0} \left(y \in x \rightarrow \bigvee_{z \in x} (y = z \cup \{z\}) \right) \right]$
$\mathrm{Function}(f) \wedge \bigvee_{y \in a} \langle y, x \rangle \in f$,) for φ, TAP produces the ZF-axioms, since the Zusatz becomes, true for each of these substitution instances. But we can formulate further axioms by chosing additional properties for φ, and FNM is therefore a proper extension of ZF. Furthermore it remains to be proved that the Axiom of Foundation (Fund) (which seems to be derivable from FNM) can be added (consistently) to the basic system. It should also be investigated whether the Axiom of Choice (AC), or possibly the Axiom of Dependent Choices (DC), and/or the Axiom of Constructability $\mathbb{V}_{FNM} = \mathcal{L}_{FNM}$ should be added to FNM or not.